Asymptotic normality of the Nadaraya-Watson estimator for non-stationary functional data and applications to telecommunications.

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Abstract

We study a non-parametric regression model where the explanatory variable is a non-stationary dependent functional data and the response variable is scalar. Supposing that the explanatory variable is a non-stationary mixture of stationary processes and general conditions of dependence of the observations (implied in particular by weakly dependence), we obtain the asymptotic normality of the Nadaraya-Watson estimator. Under some additional regularity assumptions on the regression function, it can be obtained asymptotic confidence intervals for the regression function. We apply this result to the estimation of the quality of service on the Internet where the cross traffic is non-stationary.

Keywords: Non-parametric regression, functional data, asymptotic normality, non-stationarity, dependence, quality of service.

AMS subject classification: 62G05, 62G08, 62G20.

1 Introduction

We study the problem of estimating a non-parametric regression function ϕ when the explanatory variable is functional and may be non-stationary and dependent. We observe $\{X, Y\} = \{(X_i, Y_i) : i = 0, \ldots, n-1\}$ such that

$$Y_i = \phi(X_i) + \varepsilon_i, \quad i = 0, \ldots, n-1, \tag{1}$$

where ϕ is a function $\phi : \mathcal{D} \to \mathbb{R}$, $X_i \in \mathcal{D}$, $Y_i \in \mathbb{R}$, the ε_i's are centered and independent of the X_i's and $\mathcal{D}$ is a semi-normed linear space with semi-norm $\|\cdot\|$.

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Our aim is to find the asymptotic distribution of the estimator

$$\widehat{\phi}_n(x) = \frac{\sum_{i=1}^{n-1} Y_i K\left(\frac{||x-X_i||}{h_n}\right)}{\sum_{i=1}^{n-1} K\left(\frac{||x-X_i||}{h_n}\right)} \tag{2}$$

for $x \in \mathcal{D}$ where the kernel K is a positive function and the sequence $h_n > 0$ is the bandwidth. We use the convention $0/0 = 0$.

The estimator (2) is a generalization of the Nadaraya-Watson estimator (cf. [1]). The asymptotic normality of the Nadaraya-Watson estimator is proved in [2] when the observations (X_i, Y_i) are independent, and for dependent observations, in [3].

Ferraty, Goia and Vieu ([4], [5]) proposed the estimator (2) to estimate the regression function when the explanatory variable is functional. For weakly dependent and stationary observations, they proved the complete convergence of (2) and obtained rates of convergence when the function ϕ satisfies some regularity conditions. Aspirot et al. (2005) ([6]) have generalized the result of complete convergence for the estimator (2) to a non-stationary case. Masry (2005) ([7]) has proved the asymptotic normality of $\widehat{\phi}_n$ for stationary weakly dependent random variables. Ferraty, Mas and Vieu (2007) ([8]) consider asymptotic normality of the estimator $\widehat{\phi}_n$ in the case of independence from theoretical and practical point of view. Related work on density estimation with functional data can be found in [9] and [10] for mode estimation and in [11] for asymptotic normality of kernel estimators. Surveys on functional data analysis can be found in Ramsay and Silverman (2005) ([12]) and in Ferraty and Vieu (2006) ([13]) for the non-parametric case.

Our goal in this article is to generalize the result of [7] to the case when X is a non-stationary mixture of stationary processes. More precisely, we suppose that there exist two random processes $\xi = \{\xi_n : n \in \mathbb{N}\}$, $Z = \{Z_n : n \in \mathbb{N}\}$ and a function $\varphi : \mathcal{D} \times \mathbb{R} \to \mathcal{D}$ such that ξ is stationary with values in $\mathcal{D}$, ξ is independent of Z, Z takes values in a finite set $\{z_1, \ldots, z_m\}$ of $\mathbb{R}$ and X satisfies

$$X_i = \varphi(\xi_i, Z_i), \quad i = 0, \ldots, n-1. \tag{3}$$

We suppose general conditions of dependence (cf. assumptions in Subsection 3.1), implied in particular by weakly dependence of the observations and of the process ξ, but the process Z may be non-stationary and non-weakly dependent. In Theorem 1, under these conditions and a set of technical assumptions, we obtain the asymptotic normality of the estimator (2). This result is based on central limit theorems for fields indexed in $\mathbb{N}$ or $\mathbb{Z}^d$. In Proposition 1 and 2, we prove results of central limit theorems for triangular arrays of a $\mathbb{R}^m$ -valued centered stationary random field indexed in $\mathbb{N}$ or $\mathbb{Z}^d$. For these fields the existence of a central limit theorem depends strongly on the geometry of the subset of $\mathbb{Z}^d$ where the observations are indexed. We apply these results to our model, in which we have a mixture of stationary fields $X = \varphi(\xi, Z)$ where Z is non-stationary but $X = \varphi(\xi, Z)$ conditioned to Z is stationary. We consider here our observations indexed in the level sets of Z. Some assumptions about the field Z should be made in order to ensure properties of the level sets and to have a central limit theorem in this case. In theorem 2, supposing additionally some regularity assumptions on the regression function ϕ and on the bandwidth h_n, we obtain the asymptotic normality of $\widehat{\phi}_n(x) - \phi(x)$ for $x \in \mathcal{D}$, which allows us to construct asymptotic confidence interval for $\phi(x)$.

The non-stationary assumption for the data appears naturally when modelling the Internet traffic (cf. [14], [15], [16]). In this paper, we are interested in estimating the quality of service for voice and video applications. We call quality of service parameters some network characteristics like delay, packet loss, etc.. The interest in voice or video applications is based in the fact that these network traffic (the same as many real time applications) have high quality of service requirements (very low delay and losses, etc.). In Section 4, we use the model (3) to estimate quality of service parameters for traffic over the Internet. We assume that these parameters are a function of the Internet traffic, represented by $X = \varphi(\xi, Z)$. In the literature, it is usual to assume that the traffic is piecewise stationary, in this model Z is a non-stationary random process that selects between different stationary behaviors. The process Z indicates for example if the network is very much loaded or not and it could describe seasonal or periodic behaviors. The process ξ is related with obtained measurements variations. We give simulations of the estimator (2) and confidence interval, for simulated and real traffic on the Internet.

The organization of the paper is as follows. In Section 2, we give some preliminaries on the notions of asymptotically measurable sets and fields and state a central limit theorem for triangular arrays of $\mathbb{R}^m$-valued centered stationary random fields. In Section 3, we give our result on the asymptotic normality of the estimator $\widehat{\phi}_n$ and we make the assumptions under which we obtain this result. Section 4 describes the application of our result to the problem of estimating the quality of service on the Internet. Section 5 is devoted to the proofs.

2 Preliminaries

In this section we present some preliminary results. In subsection 2.1 we define the notion of asymptotically measurable subsets that allows, in subsection 2.2, to give a central limit theorem for triangular arrays of $\mathbb{R}^m$-valued centered stationary random fields. In subsection 2.3 we present the notion of asymptotically measurable field that allows to state a central limit theorem for a field described by equation (3). In this section, we consider both the case when the field is indexed in $\mathbb{N}$ and in $\mathbb{Z}^d$ and in the following, $\mathbb{L}$ stands for $\mathbb{N}$ or $\mathbb{Z}^d$.

When we have $\mathbb{L} = \mathbb{N}$, we denote $\ell(n) = n$ and for a set $A \subset \mathbb{N}$, $A_n = A \cap \{0, 1, \ldots, n-1\}$ and $A_n^c = A \cap \{0, 1, \ldots, n-1\}$.

When we have $\mathbb{L} = \mathbb{Z}^d$, we denote $\ell(n) = (2n+1)^d$ and for a set $A \subset \mathbb{Z}^d$, $A_n = A \cap \{-n, \ldots, n\}^d$ and $A_n^c = A^c \cap \{-n, \ldots, n\}^d$.

2.1 Notion of asymptotically measurable subsets

In what follows $card(A)$ indicates the cardinal of subset A.

Definition 1. *A subset $A \subset \mathbb{L}$ is said to be an asymptotically measurable subset if, for all $k \in \mathbb{L}$, the following limit exists*

$$F(k, A) = \lim_{n \to \infty} \frac{card\{A_n^c \cap (A_n - k)\}}{\ell(n)}.$$

The function $F(., A)$ is called the border function of the subset A.

Definition 2. *The subset family $\{A^i : i = 1, \ldots, m\}$ in $\mathbb{L}$ is an asymptotically measurable family if, for all $i \in \{1, \ldots, m\}$, the subsets A^i are asymptotically measurable and, for all $k \in \mathbb{L}$ and*

$i, j \in \{1, \dots, m\}$, *the following limit exists*

$$F\left(k, A^i, A^j\right) = \lim_{n\to\infty} \frac{card\{A_n^i \cap (A_n^j - k)\}}{\ell(n)}.$$

2.2 Central limit theorem for triangular arrays of $\mathbb{R}^m$-valued centered stationary random field

In the following, the notation $\stackrel{w}{\Longrightarrow}$ means convergence in law and $N_d(0,\Sigma)$ represents a zero mean normal distribution in dimension d with covariance matrix Σ. Moreover, for $X^n = (X_k^n)_{k\in\mathbb{L}}$ a $\mathbb{R}^m$-valued centered stationary random field, we denote the components of X^n by $X^n = (X^{1,n}, \dots, X^{m,n})$ and $X_k^n = (X_k^{1,n}, \dots, X_k^{m,n})$ for $k \in \mathbb{L}$.

Definition 3. *We define the class $B(\mathbb{L})$ as the class of $\mathbb{R}^m$- valued centered stationary random fields $X^n = (X_k^n)_{k\in\mathbb{L}}$ that satisfies the following conditions:*

(H_1) For all $i, j \in \{1, \dots, m\}$ and $n \in \mathbb{N}$, we have

$$\sum_{k\in\mathbb{L}} \left| E\left\{X_0^{i,n} X_k^{j,n}\right\}\right| < \infty.$$

(H_2) Let $X^{n,J} = (X^{1,n,J}, \dots, X^{m,n,J})$ be the truncation by J of X^n, defined for $k \in \mathbb{L}$ and $J > 0$ by

$$X_k^{n,J} = X_k^n 1_{\{\|X_k^n\|\le J\}} - E\left[X_k^n 1_{\{\|X_k^n\|\le J\}}\right],$$

where $\|\cdot\|$ represents the euclidian norm on $\mathbb{R}^m$.

(i) There exists a sequence $\gamma(k) \ge 0$ such that $\sum_{k\in\mathbb{L}} \gamma(k) < \infty$ and such that for all $k \in \mathbb{L}$, $n \in \mathbb{N}$, $i, j \in \{1, \dots, m\}$ and $J > 0$ we have

$$\left| E\left\{X_0^{i,n,J} X_k^{j,n,J}\right\}\right| \le \gamma(k).$$

(ii) There exists a sequence $b(J)$ such that $\lim_{J\to\infty} b(J) = 0$ and for all set $B \subset \mathbb{L}$, $i \in \{1, \dots, m\}$, $n \in \mathbb{N}$ and $J > 0$

$$E\left[\left(S_n\left(B, X^{i,n}\right) - S_n\left(B, X^{i,n,J}\right)\right)^2\right] \le \frac{b(J)card(B_n)}{\ell(n)},$$

where

$$S_n(B, X^{i,n}) = \frac{1}{(\ell(n))^{1/2}} \sum_{k\in B_n} X_k^{i,n}.$$

(H_3) There exists a sequence of reals numbers $C(J)$ such that for all $B \subset \mathbb{L}$, $n \in \mathbb{N}$, $J > 0$ and $i \in \{1, \dots, m\}$ we have

$$E\left[S_n\left(B, X^{i,n,J}\right)^4\right] \le C(J)\left(\frac{card(B_n)}{\ell(n)}\right)^2.$$

(H_4) There exist a function $h : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\lim_{x\to+\infty} h(x) = 0$ and a function $g : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}_+$, with $g(J,t) < \infty$ for all fixed $J > 0$ and $\sup_{t\in\mathbb{R}^m} g(J,t) = g_J < \infty$, such that

$$\left| E\left[e^{iS_n(B\cup C,\langle t,X^{n,J}\rangle)}\right] - E\left[e^{iS_n(B,\langle t,X^{n,J}\rangle)}\right] E\left[e^{iS_n(C,\langle t,X^{n,J}\rangle)}\right]\right| \leq g(J,t)h(d(B,C)),$$

for all disjoint sets $B, C \subset \mathbb{L}$, all $n \in \mathbb{N}$, $J > 0$ and $t \in \mathbb{R}^m$. Here $\langle\cdot,\cdot\rangle$ represents the scalar product on $\mathbb{R}^m$.

(H_5) There exist sequences $\gamma^J(i,j,k)$ and $\gamma(i,j,k)$, $i, j \in \{1,\ldots,m\}$, $k \in \mathbb{L}$ such that for all $J > 0$ we have

$$\lim_{n\to\infty} E\left\{X_0^{i,n,J} X_k^{j,n,J}\right\} = \gamma^J(i,j,k),$$

and

$$\lim_{J\to\infty} \gamma^J(i,j,k) = \gamma(i,j,k) \tag{4}$$

for $i, j \in \{1,\ldots,m\}$ and $k \in \mathbb{L}$.

We have the two following propositions proved in Section 5.

Proposition 1. *If $X^n = (X_k^n)_{k\in\mathbb{N}}$ belongs to $B(\mathbb{N})$, then for any asymptotically measurable family $\left\{A^i : i = 1,\ldots,m\right\}$ in $\mathbb{N}$, we have as n tends to ∞*

$$(S_n(A^1, X^{1,n}),\ldots,S_n(A^m, X^{m,n})) \stackrel{w}{\Longrightarrow} N_m(0,\Sigma),$$

where for $i, j \in \{1,\ldots,m\}$

$$\Sigma(i,j) = \gamma(i,j,0)F(i,j,0) + \sum_{k\geq 1}\{\gamma(i,j,k)F(i,j,k) + \gamma(j,i,k)F(j,i,k)\}$$

and

$$F(i,j,k) = \lim_{n\to\infty} \frac{card\{A_n^i \cap (A_n^j - k)\}}{n}.$$

Proposition 2. *If $X^n = (X_k^n)_{k\in\mathbb{Z}^d}$ belongs to $B(\mathbb{Z}^d)$, then for any asymptotically measurable family $\left\{A^i : i = 1,\ldots,m\right\}$ in $\mathbb{Z}^d$, we have as n tends to ∞*

$$(S_n(A^1, X^{1,n}),\ldots,S_n(A^m, X^{m,n})) \stackrel{w}{\Longrightarrow} N_m(0,\Sigma),$$

where for $i, j \in \{1,\ldots,m\}$

$$\Sigma(i,j) = \sum_{k\in\mathbb{L}} \gamma(i,j,k)F(i,j,k)$$

and

$$F(i,j,k) = \lim_{n\to\infty} \frac{card\{A_n^i \cap (A_n^j - k)\}}{(2n+1)^d}.$$

2.3 Notion of asymptotically measurable field

In what follows $\xrightarrow[n]{a.s.}$ means almost sure convergence as n tends to ∞ and 1_A is the indicator function of a set A. In the case $\mathbb{L} = \mathbb{N}$, we set $D^{(n)} = \{0, \dots, n-1\}$ and if $\mathbb{L} = \mathbb{Z}^d$, we set $D^{(n)} = \{-n, \dots, n\}^d$.

Definition 4. *The $\mathbb{R}^m$-valued random field $Z = (Z_n)_{n\in\mathbb{L}}$ is an asymptotically measurable field in $\mathbb{L}$ if there exists a random probability measure R_0 in $\mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra in $\mathbb{R}^m$, such that*

$$\frac{1}{\ell(n)} \sum_{m\in D^{(n)}} 1_{\{Z_m\in B\}} \xrightarrow[n]{a.s.} R_0(B)$$

and for all $k \in \mathbb{L} - \{0\}$ there exists a random measure R_k in $\mathcal{B}_2$, where $\mathcal{B}_2$ is the Borel σ-algebra in $\mathbb{R}^{2m}$, such that for all $B, C \in \mathcal{B}$

$$\frac{1}{\ell(n)} \sum_{m\in D^{(n)}} 1_{\{Z_m\in B\}} 1_{\{Z_{m-k}\in C\}} \xrightarrow[n]{a.s.} R_k(B \times C).$$

If the limit measure is non-random Z is called a regular field in $\mathbb{L}$.

The following proposition (cf. Proof in [17]) relates asymptotically measurable and regular fields with asymptotically measurable subsets.

Proposition 3. *If $Z = (Z_n)_{n\in\mathbb{L}}$ is an asymptotically measurable field, $B^1, B^2, \dots, B^m$ are disjoint subsets of $\mathcal{B}$ and $A^i = \{n \in \mathbb{L} : Z_n \in B^i\}$ for $i = 1, \dots, m$, then, conditionally to Z the family $\{A^1, \dots, A^m\}$ is an asymptotically measurable family with $F(k, A^i, A^j) = R_k(B^i, B^j)$ and $F(0, A^i, A^j) = R_0(B^i)\delta_{ij}$, where $\delta_i j$ is the Kronecker delta.*

Remarks: In the next sections we will assume further hypotheses on Z for the field described in (3) in order to prove the results, but the previous proposition is the tool for conditioning on level sets of the field Z and applying the results for stationary fields to the non-stationary case (3).

3 Results

In the following, we consider a fixed $x \in \mathcal{D}$ and we will study the asymptotic normality of $\widehat{\phi}_n(x)$ given by (2). Proofs of the results are in Section 5.

The estimator $\widehat{\phi}_n$ can be also written

$$\widehat{\phi}_n(x) = \frac{g_n(x)}{f_n(x)},$$

where

$$g_n(x) = \frac{1}{n\psi(h_n)} \sum_{i=0}^{n-1} Y_i K_n(X_i),$$

$$f_n(x) = \frac{1}{n\psi(h_n)} \sum_{i=0}^{n-1} K_n(X_i) \tag{5}$$

with

$$K_n(u) = K\left(\frac{\|u-x\|}{h_n}\right), \quad u \in \mathcal{D}. \tag{6}$$

and $\psi(h_n)$ is given in assumption (A_1). We recall that we use the convention $0/0 = 0$.

In Subsection 3.1, we give the assumptions under which the asymptotic normality of $\widehat{\phi}_n(x)$ is obtained and in Subsection 3.2 we give our results.

3.1 Assumptions

We make the following assumptions on the distribution of (X, Y).

(A_1) There exist positive functions $\psi, \psi_1, \dots, \psi_m$ defined on $\mathbb{R}_+ \times \mathcal{D}$, functions $c_1, \dots, c_k$ defined on $\mathcal{D}$ and a subset Δ of $\{1, \dots, m\}$ such that for all $h > 0$

$$P\left[\|\varphi(\xi_1, z_k) - x\| \le h\right] = c_k(x)\psi_k(h, x),$$

where $\lim_{h\to 0} \dfrac{\psi_k(h,x)}{\psi(h,x)} = 1$ if $k \in \Delta$, and $\lim_{h\to 0} \dfrac{\psi_k(h,x)}{\psi(h,x)} = 0$ if $k \in \Delta^C$.
In the following, to simplify the notation, we replace $\psi(h_n, x)$ by $\psi(h_n)$, but the quantity $\psi(h_n)$ still depends of x.

(A_2) The functions $u \mapsto \psi_k(u, x)$ are differentiable on $\mathbb{R}_+$, with differential function denoted $\psi_k'(u, x)$ and satisfy

$$\lim_{h\to 0} \frac{h}{\psi_k(h,x)} \int_0^1 K(u)\psi_k'(uh, x)du = d_k(x),$$

where the d_k's are functions defined on $\mathcal{D}$.

(A_3) The process Z is regular and satisfies for $k \in \{1, \dots, m\}$ as n tends to ∞ that

$$\sqrt{n\psi(h_n)}\left(\frac{1}{n}\sum_{i=0}^{n-1} 1_{\{Z_i = z_k\}} - \frac{1}{n}\sum_{i=0}^{n-1} P(Z_i = z_k)\right) \stackrel{P}{\Longrightarrow} 0, \tag{7}$$

where $\stackrel{P}{\Longrightarrow}$ means convergence in probability. For $k \in \{1, \dots, m\}$, denote by p_k the limit

$$p_k = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} P(Z_i = z_k).$$

Let the $\mathbb{R}^{2m}$-value random process $\tilde{X}^n = (\tilde{X}^{1,n}, \dots, \tilde{X}^{2m,n})$ defined for $i \in \mathbb{N}$ as follows:

- for $l \in \{1, \dots, m\}$

$$\begin{aligned}\tilde{X}_i^{l,n} &= \frac{1}{\sqrt{\psi(h_n)}} K_n\left(\varphi(\xi_i, z_l)\right)\left(\phi(\varphi(\xi_i, z_l)) + \varepsilon_i\right) \\ &\quad - \frac{1}{\sqrt{\psi(h_n)}} E\left[K_n\left(\varphi(\xi_i, z_l)\right)\phi(\varphi(\xi_i, z_l))\right],\end{aligned}$$

- for $l \in \{m+1, \ldots, 2m\}$

$$\tilde{X}_i^{l,n} = \frac{1}{\sqrt{\psi(h_n)}} K_n\left(\varphi(\xi_i, z_l)\right) - \frac{1}{\sqrt{\psi(h_n)}} E\left[K_n\left(\varphi(\xi_i, z_l)\right)\right],$$

  where K_n is defined by (6).

Moreover we assume the following hypotheses.

(A_4) The $\mathbb{R}^{2m}$-value random process $\tilde{X}^n$ belongs to $B(\mathbb{N})$.

(A_5) The function ϕ is continuous.

(A_6) The estimator $f_n(x)$ defined by (5) converges in probability to $f(x) > 0$ where the function f is defined by

$$f(u) = \sum_{k \in \Delta} p_k d_k(u) c_k(u), \quad u \in \mathcal{D}.$$

We suppose the following hypothesis on the estimator $\widehat{\phi}_n$.

(A_7) The function K is a positive function with support $[0, 1]$.

(A_8) The bandwidth h_n satisfies

$$\lim_{n \to \infty} h_n = 0$$

and

$$\lim_{n \to \infty} n\psi(h_n) = \infty.$$

Remarks on the assumptions:

- The conditions (A_1) and (A_2) mean that, conditionally on Z, the distance between the observation X_i and the estimation point x has a density. The quantity $\psi_k(h, x)$ plays a role analogous to h^d for multidimensional estimation.

- The hypothesis (A_3) means that the variable Z does not behave too irregularly. It is assumed that the process has some kind of "stationarity in mean". However this is not a strong assumption, and for example (A_3) could be satisfied by periodic or semi-periodic random variables. The relation (7) is satisfied for example if the variables $1_{Z_i = z_k}$ satisfy a central limit theorem for each $k = 1, \ldots, m$.

- The hypothesis (A_4) can be replaced by various sets of hypothesis doing a trade-off between conditions on the moments of Y and conditions of mixing. Perera (1997) ([18]) describes several sets of conditions that imply the previous hypotheses.

- In the hypothesis (A_6), $f(x) > 0$ means roughly that there is a positive probability of finding observations X_i near (in the sense of the semi-norm on $\mathcal{D}$) the point x where we want to compute the estimator." The function f play the role of the density of the whole process $X = \varphi(\xi, Z)$. Moreover the convergence in probability can be obtained under the conditions (A_1) y (A_2), conditions on the moments of Y_i and conditions of mixing (cf. Proof in [6]).

- The hypothesis (A_5), (A_7) and (A_8) are classical for obtaining asymptotic normality.

3.2 Asymptotic normality of $\widehat{\phi}_n(x)$

The asymptotic normality of $\widehat{\phi}_n(x)$ is obtained in three stages: asymptotic normality of the field $\tilde{X}^n$ in Proposition 4, of $(g_n(x) - E(g_n(x)), f_n(x) - E(f_n(x)))$ in Proposition 5 and then of $\widehat{\phi}_n(x)$ in Theorem 1. Theorem 2 gives the asymptotic normality of $\widehat{\phi}_n(x) - \phi(x)$ which allows to construct asymptotic confidence interval for $\phi(x)$.

Proposition 4. *For $l \in \{1, \ldots, m\}$, denote $A^l = \{k \in \mathbb{N},\ Z_k = z_l\}$ and $A^{l+m} = A^l$. If Z is asymptotically measurable, then under Assumption (A_4), conditionally on Z, we have, as n tends to ∞*

$$(S_n(A^1, \tilde{X}^{1,n}), \ldots, S_n(A^{2m}, \tilde{X}^{2m,n})) \overset{w}{\Longrightarrow} N_{2m}(0, \Sigma),$$

where for $i, j \in \{1, \ldots, 2m\}$

$$\Sigma(i,j) = \gamma(i,j,0)F(i,j,0) + \sum_{k \in \mathbb{N}} \left\{\gamma(i,j,k)F(i,j,k) + \gamma(j,i,k)F(j,i,k)\right\},$$

with $\gamma(i,j,k)$, $k \in \mathbb{N}$, defined by (4) for the field $\tilde{X}^n$ and

$$F(i,j,k) = \lim_{n \to \infty} \frac{card\{A_n^i \cap (A_n^j - k)\}}{n}.$$

Proposition 5. *Under Assumptions (A_3) and (A_4), we have, as n tends to ∞, that*

$$\sqrt{n\psi(h_n)}\Big(g_n(x) - E(g_n(x)), f_n(x) - E(f_n(x))\Big) \overset{w}{\Longrightarrow} N_2(0, A),$$

where

$$A = \begin{pmatrix} a_1(x) & a_2(x) \\ a_2(x) & a_3(x) \end{pmatrix}$$

with

$$a_1(x) = \sum_{i,j=1}^{m} \Sigma(i,j), \quad a_2(x) = \sum_{i=1}^{m} \sum_{j=m+1}^{2m} \Sigma(i,j), \quad a_3(x) = \sum_{i,j=m+1}^{2m} \Sigma(i,j).$$

Theorem 1. *Under Assumptions (A_1)–(A_8), we have as n tends to ∞*

$$\sqrt{n\psi(h_n)}\left(\widehat{\phi}_n(x) - \frac{E(g_n(x))}{E(f_n(x))}\right) \overset{w}{\Longrightarrow} N_1(0, \sigma^2(x)),$$

where

$$\sigma^2(x) = \frac{a_1(x) - 2a_2(x)\phi(x) + a_3(x)\phi^2(x)}{f^2(x)}. \tag{8}$$

Theorem 2. *We suppose that*

(B_1) the function ϕ satisfies

$$|\phi(u) - \phi(v)| \leq L\|u - v\|^{\beta},$$

with $\beta > 0$ and L a positive constant,

(B_2) there exist $M > 0$ and a function α such that for each $k \in \{1, \ldots, m\}$ and $h > 0$, we have

$$\left| \frac{P\left[\|\varphi(\xi_1, z_k) - x\| \leq h\right]}{\psi_k(h, x)} - c_k(x) \right| \leq M\alpha(h) \quad \forall \alpha > 0.$$

(B_3) the bandwidth h_n satisfies

$$\lim_{n\to\infty} \alpha(h_n)\sqrt{n\psi(h_n)} = 0$$

and

$$\lim_{n\to\infty} h_n^{\beta}\sqrt{n\psi(h_n)} = 0$$

(B_4) for each $k \in \{1, \ldots, m\}$, we have

$$\lim_{n\to\infty} \sqrt{n\psi(h_n)} \left(\frac{1}{n} \sum_{i=0}^{n-1} P(Z_i = z_k) - p_k \right) = 0.$$

Then under Assumptions (A_1)–(A_8), (B_1)–(B_4), we have as n tends to ∞,

$$\sqrt{n\psi(h_n)} \left(\widehat{\phi}_n(x) - \phi(x) \right) \overset{w}{\Longrightarrow} N_1(0, \sigma^2(x)),$$

where $\sigma^2(x)$ is defined by (8).

4 Application to end-to-end quality of service estimation

4.1 Motivation and model

Nowadays new services are offered over the Internet like voice and video. For these new applications the need to measure the network performance has increased. Multimedia applications have several requirements in terms of delay, losses and other quality of service parameters. These constraints are stronger than the ones for usual data transfer applications (mail, etc.). Measurements of the Internet performance are necessary for different reasons, for example to advance in understanding the behavior of the Internet or to verify the quality of service assured to the new services.

There are several measuring techniques, most of them with software implementations, that can be classified in active (sending controlled traffic called probe packets) and passive measures (generally measures at the routers). The aim of these techniques is to detect different characteristics of the network that can be the topology or some performance parameters (delay, losses, available bandwidth, etc.). There are also several problems to measure the performance parameters, for example the route of the packets can change, the traffic bit rate is not constant and normally is in bursts, the probe packets can be filtered or altered by one ISP (Internet Service Provider) in the path, there is not clock synchronization between routers and end equipments, etc.. Normally the internal routers in the path between two points of interest are not under the control of only one user or one ISP. Therefore, it is not very useful to have measuring procedures that depend on the information of the internal routers. For this reason end-to-end measures is one of the most developed methodologies during the last years (cf. [19], [20],[21]).

We consider a single link and a voice or video traffic that we want to monitor, that is we want to know the quality of service parameters when this traffic goes between two points over the network. The link is shared by this multimedia traffic and many other traffics in the network that are unknown and that is called cross traffic and could be modelled as a stochastic process. The performance parameter for packets of a multimedia traffic could be delay, packets losses, delay variation (or all of them together) and we will consider it as a random variable Y. This performance parameter could be modeled as a function of the cross traffic stochastic process, the video or voice stochastic process, the link capacity and the buffer size. We want to estimate Y without sending video or voice traffic during long time periods. Active measurements, that send traffic, are useful if they do not charge the network, so they consist on small probe packets sent during short time periods.

The link capacity and the buffer size are not known but it is assumed that they are constants during the monitoring process. The multimedia traffic is also known therefore we can suppose that $Y = \phi(X)$ where X is a characterization of the cross traffic stochastic process. We can not measure the cross traffic process but we can obtain data closely related with this process. This data is described in the following subsection. The cross traffic process on the Internet is a dependent non-stationary process (c.f [14] or [15]). To take into account non-stationary behaviors we will suppose that X satisfies the relation (3).

4.2 Measurement procedure

In what follows we describe the procedure to estimate the function ϕ. We divide the experiment in two phases: first, we send a burst of small probe packets (pp) of fixed size spaced by a fixed time. Immediately after the burst we send during a short time a video stream. We repeat the previous procedure during some time lapse, sending a new burst and a video probe after a time interval measured from the previous end of the video stream as it is shown in the scheme in figure 1.

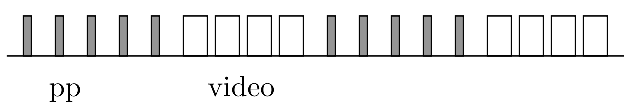

Figure 1: Measurement procedure scheme

With the probe packets burst we infer the cross traffic of the link. We measure at the output of the link the interarrival time between consecutive probe packets. This time serie is strongly correlated with the cross traffic process that shares the link with the probe traffic. We compute the empirical distribution function of probe packets interarrival times because this distribution is related with the queue behavior in the network. Using this cross traffic estimation and measuring the delay Y over the video, we want to estimate the function ϕ.

From each probe packet burst and video sequence j we have a pair (X_j, Y_j), where X_j is the empirical distribution function of interarrival times and Y_j is the performance metric of interest measured from the video stream j.

Therefore, our estimation problem has been transformed into the problem of inferring a function $\phi : \mathcal{D} \to \mathbb{R}$ where $\mathcal{D}$ is the space of the probability distribution functions and $\mathbb{R}$ is the real line such that the (X_j, Y_j) satisfy (1).

We did the estimations both with simulated and real traffic. Real data are obtained with a software that do the active measures described before between a server and its users. For simulated data, the variable Y is the one-way delay which is the delay between two points. For real data, we measure round trip time (RTT) delay that is the delay in the whole trip between two points in a path and in the reverse path.

For simulated traffic, we have 360 observations (X_j, Y_j). For each $j \in \{1, \ldots, 359\}$, we calculate the estimator $\widehat{\phi}_j$ defined by (2) using the observations $(X_1, Y_1), \ldots, (X_j, Y_j)$ and we represent on figures 2 and 3 the values of Y_{j+1} and of its prediction $\widehat{\phi}_j(X_{j+1})$. Moreover for each $j \in \{1, \ldots, 359\}$, we calculate a confidence interval around $\widehat{\phi}_j(X_{j+1})$ using a block-bootstrap. The estimator $\widehat{\phi}_j$ is calculated using the kernel $K(x) = (x^2 - 1)^2 1_{[-1,1]}(x)$, the bandwidth that minimizes the sum of the relative error in the estimation before the time j and the L_1 norm. The figure 4 represents the relative error of the estimation. The results are relatively accurate for sample of size larger than 100.

For real data, we have 50 observations (X_j, Y_j) and we realize the same type of estimation. In figure 5, we represents for each $j \in \{1, \ldots, 49\}$, the values of Y_{j+1} and of its prediction $\widehat{\phi}_j(X_{j+1})$ and a confidence a confidence band around $\widehat{\phi}_j(X_{j+1})$ using a block-bootstrap. The figure 6 represents the relative error of the estimation. The results are less accurate than in the case of simulated data but the estimations are calculated with samples of small size.

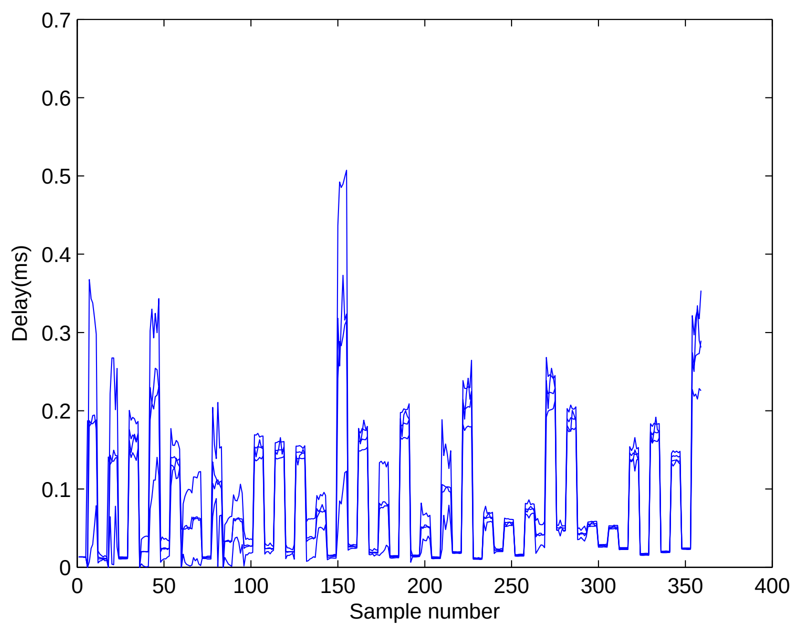

Figure 2: Estimated delays for all simulated data

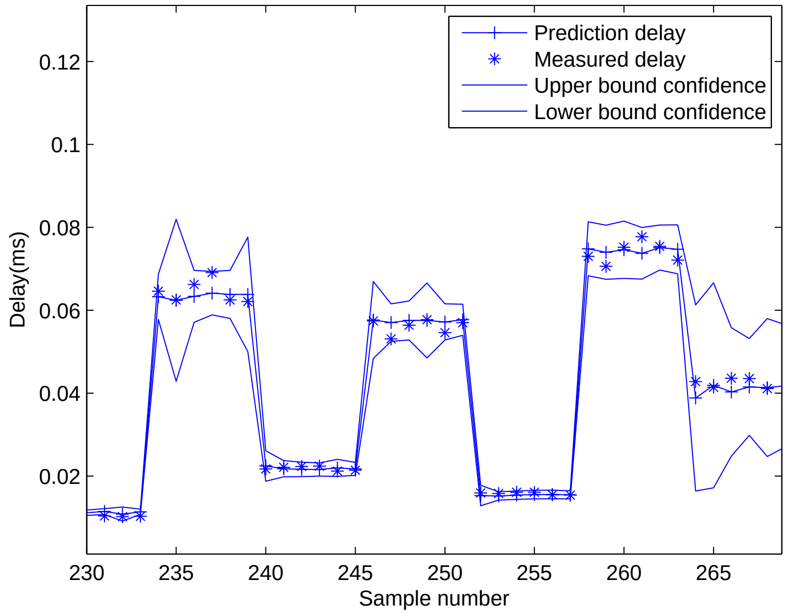

Figure 3: Estimated delays for simulated data observations between 230 and 270

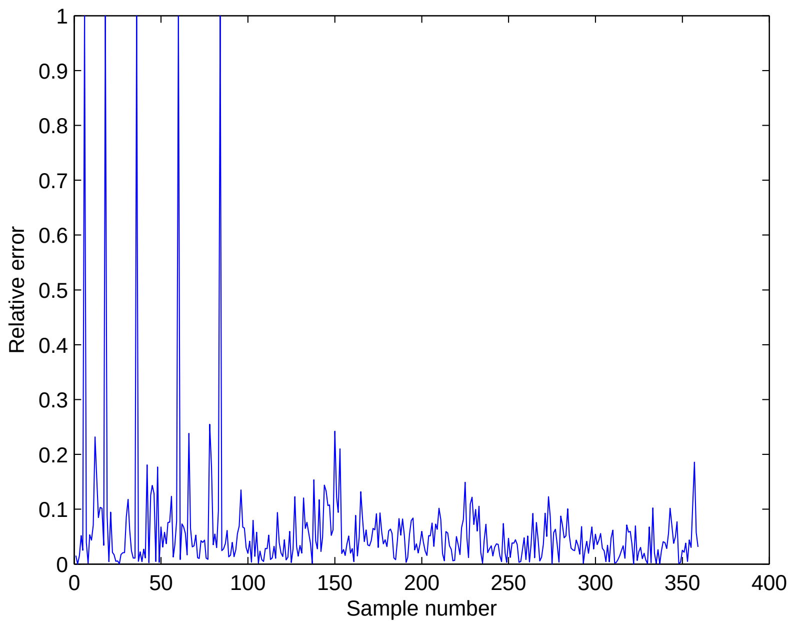

Figure 4: Relative error for simulated data

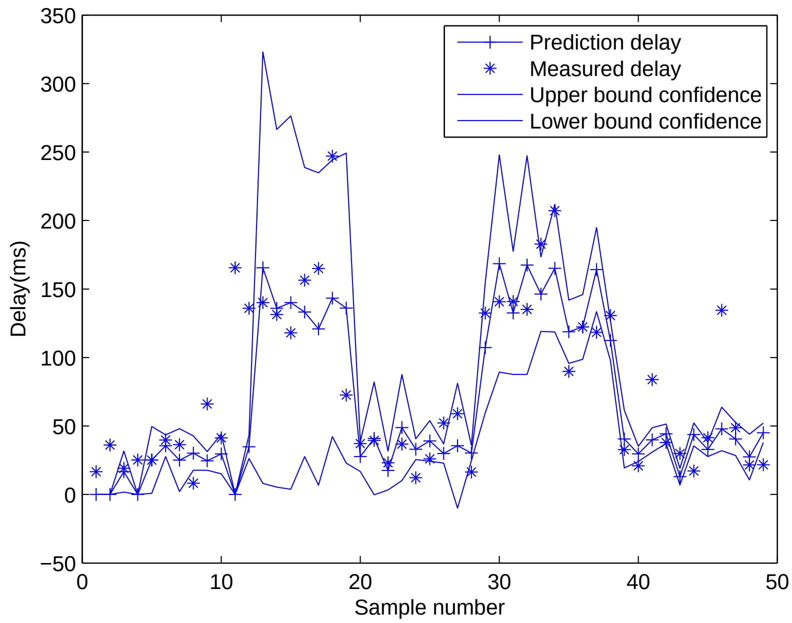

Figure 5: Estimated RTT for real data

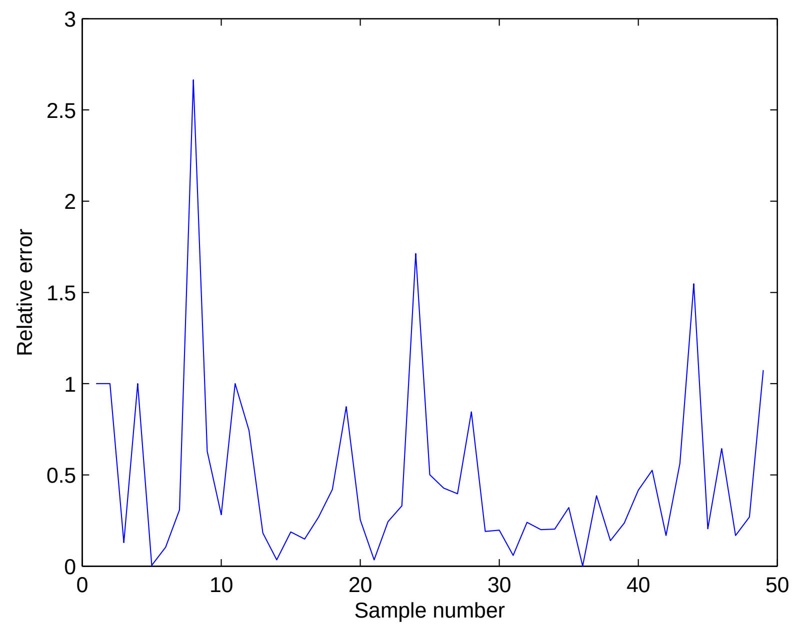

Figure 6: Relative error for real data

5 Proof of the propositions and theorems

5.1 Proof of Propositions 1 and 2

This result is obtained by means of Bernshtein method (cf. [18], [22]) and it is a generalization of the theorem 4.15 on triangular arrays for $\mathbb{R}$-valued random fields obtained by Tablar (2006) ([23], p 86) to $\mathbb{R}^m$-valued random fields. To prove that the vectorial field $(S_n(A^1, X^{1,n}), \ldots, S_n(A^m, X^{m,n})) \overset{w}{\Longrightarrow} N_m(0, \Sigma)$, it is sufficient to prove that for all $\lambda \in \mathbb{R}^m$ $\langle \lambda, S_n \rangle \overset{w}{\Longrightarrow} N(0, \lambda^t \Sigma \lambda)$ where $S_n = (S_n(A^1, X^{1,n}), \ldots, S_n(A^m, X^{m,n}))$ and $\lambda^t = (\lambda_1, \ldots, \lambda_m)$. We have that

$$\begin{aligned} \langle S_n, \lambda \rangle &= \sum_{i=1}^{m} \lambda_i S_n(A^i, X^{i,n}) = \sum_{i=1}^{m} \frac{1}{(\ell(n))^{1/2}} \lambda_i \sum_{k \in A_n^i} X_k^{i,n} \\ &= \frac{1}{(\ell(n))^{1/2}} \sum_{k \in D^{(n)}} \sum_{i=1}^{m} \lambda_i X_k^{i,n} 1_{\{k \in A_n^i\}} \end{aligned}$$

Let us consider a real valued field $X^n(\lambda)$ where $X_k^{i,n}(\lambda) = \lambda_i X_k^{i,n}$ if $k \in A_n^i$. Then $\langle S_n, \lambda \rangle = S_n\left(D^{(n)}, X^n(\lambda)\right)$. As the field $X^n(\lambda)$ verifies the hypotheses in the work of [23] we can apply the theorem for a real valued random field in order to obtain the result of propositions 1 and 2.

5.2 Proof of Proposition 4

Since Z is asymptotically measurable, conditionally to Z, the subset family $(A^1, \ldots, A^{2m})$ is an asymptotically measurable family in $\mathbb{N}$. Then applying Proposition 1 to $\tilde{X}$ which belongs to $B(\mathbb{N})$, we deduce the result.

5.3 Proof of Proposition 5

Before proving Proposition 5, here we prove the following lema.

Lemma 1. *Let $k \in \{1, \ldots, m\}$. Under the Assumptions (A_1), (A_2), (A_3), (A_5), (A_7) and (A_8), we have*

1) the following limit

$$\lim_{n\to\infty} \frac{1}{\psi(h_n)} E\left[K_n(\varphi(\xi_1, z_k))\right] = d_k(x) c_k(x) 1_{\{k \in \Delta\}},$$

2) for $\beta > 0$,

$$\lim_{n\to\infty} \frac{1}{\psi(h_n)} E\left[\left\|\frac{\varphi(\xi_1, z_k) - x}{h_n}\right\|^{\beta} K_n(\varphi(\xi_1, z_k))\right] \le d_k(x) c_k(x),$$

3) that the estimator $f_n(x)$ satisfies

$$\lim_{n\to\infty} E(f_n(x)) = f(x),$$

4) and the estimator $g_n(x)$ satisfies

$$\lim_{n\to\infty} E(g_n(x)) = \phi(x)f(x).$$

Proof. The results 1) and 2) of this lemma come from the fact that under Assumption (A_1), the density of $\|\varphi(\xi_1, z_k) - x\|$ is the function $u \to c_k(x)\psi_k'(u,x)$. Then under Assumption (A_7)

$$E\left[K_n(\varphi(\xi_1, z_k))\right] = h_n c_k(x) \int_0^1 K(u)\psi_k'(uh_n, x)du$$

and

$$E\left[\left\|\frac{\varphi(\xi_1, z_k) - x}{h_n}\right\|^\beta K_n(\varphi(\xi_1, z_k))\right] = h_n c_k(x) \int_0^1 K(u)u^\beta \psi_k'(uh_n, x)du$$

and it is sufficient to use Assumptions (A_2) and (A_8) to conclude.

Here we prove 3) and 4). We have for $i \in \{1, \ldots, n\}$,

$$E(K_n(X_i)) = E\{E(K_n(X_i)|Z_i)\} = \sum_{k=1}^m E\{K_n(\varphi(\xi_i, z_k))\}P(Z_i = z_k).$$

As $(\varphi(\xi_i, z_k))_{i=1,\ldots,n}$ is identically distributed, we obtain that,

$$E(f_n(x)) = \sum_{k=1}^m \left(\frac{1}{\psi(h_n)} E\left(K_n(\varphi(\xi_1, z_k))\right) \frac{1}{n}\sum_{i=0}^{n-1} P(Z_i = z_k)\right). \tag{9}$$

Now applying (A_3) and the assertion 1) of this lemma in (9), we obtain 3). Similar calculus give that

$$E(g_n(x)) = \sum_{k=1}^m \left(\frac{1}{\psi(h_n)} E\left(\phi\left(\varphi(\xi_1, z_k)\right) K_n(\varphi(\xi_1, z_k))\right) \frac{1}{n}\sum_{i=0}^{n-1} P(Z_i = z_k)\right).$$

Then we have

$$E(g_n(x)) = \phi(x)E(f_n(x)) + R_n,$$

where

$$\begin{aligned} R_n &= \sum_{k=1}^m \left(\frac{1}{\psi(h_n)} E\left(\{\phi\left(\varphi(\xi_1, z_k)\right) - \phi(x)\} K_n(\varphi(\xi_1, z_k))\right) \frac{1}{n}\sum_{i=0}^{n-1} P(Z_i = z_k)\right) \\ &\le \sup_{u:\|x-u\|\le h_n} |\phi(u) - \phi(x)|\, E(f_n(x)) \end{aligned}$$

and we obtain 4) using the continuity of function ϕ (Assumption (A_5)) and using that $\lim_{n\to\infty} E(f_n(x)) = f(x)$. □

Conditionally to Z, we have

$$\sqrt{n\psi(h_n)}\Big(g_n(x) - E(g_n(x)|Z), f_n(x) - E(f_n(x)|Z)\Big) = \left(\sum_{j=1}^m S_n(A^j, \tilde{X}_{n,j}), \sum_{j=m+1}^{2m} S_n(A^j, \tilde{X}_{n,j})\right).$$

Using Proposition 4, we obtain that, conditionally to Z,

$$\left(\sum_{j=1}^{m} S_n(A^j, \tilde{X}_{n,j}), \sum_{j=m+1}^{2m} S_n(A^j, \tilde{X}_{n,j})\right) \stackrel{w}{\Longrightarrow} N_2(0, A).$$

Since Z is regular, the matrix A is not random and then we have

$$\sqrt{n\psi(h_n)}\Big(g_n(x) - E(g_n(x)|Z), f_n(x) - E(f_n(x)|Z)\Big) \stackrel{w}{\Longrightarrow} N_2(0, A).$$

Now

$$\begin{aligned}(g_n(x) - E(g_n(x)), f_n(x) - E(f_n(x))) &= (g_n(x) - E(g_n(x)|Z), f_n(x) - E(f_n(x)|Z)) \\ &\quad + (E(g_n(x)|Z) - E(g_n(x)), E(f_n(x)|Z) - E(f_n(x))).\end{aligned}$$

We have

$$E(f_n(x)|Z) - E(f_n(x)) = \sum_{j=1}^{2m} \frac{E(K_n(\xi_1, z_j))}{\psi(h_n)} \left(\frac{card(A_n^j)}{n} - \frac{1}{n}\sum_{i=0}^{n-1} P[Z_i = j]\right).$$

Under Assumption (A_3) and using assertion 1) of Lemma 1, we have that

$$\sqrt{n\psi(h_n)}\,(E(f_n(x)|Z) - E(f_n(x)))$$

converges in probability to 0 as n tends to ∞. Using Assumption (A_3), $\sqrt{n\psi(h_n)}\,(E(g_n(x)|Z) - E(g_n(x)))$ converges in probability to 0 as n tends to ∞. Using the lemma of Slutsky, we obtain the result of the proposition.

5.4 Proof of Theorem 1

We have

$$\widehat{\phi}_n(x) - \frac{E(g_n(x))}{E(f_n(x))} = \frac{Q_n(x) - B_n(x)(f_n(x) - E\,(f_n(x)))}{f_n(x)},$$

where

$$B_n(x) = \frac{E(g_n(x))}{E(f_n(x))} - \phi(x)$$

and

$$Q_n(x) = (g_n(x) - E(g_n(x))) - \phi(x)(f_n(x) - E(f_n(x))).$$

Using 3) y 4) of Lemma 1, we deduce that

$$\lim_{n\to\infty} B_n(x) = 0.$$

Then using (A_6), we have that $f_n(x)$ converges in probability to $f(x) > 0$ and then it implies that

$$\frac{B_n(x)(f_n(x) - E\,(f_n(x)))}{f_n(x)} \tag{10}$$

converges in probability to 0.

Finally, using (A_6), (10) and Proposition 5, applying Slutsky Lemma, we deduce the result of the proposition.

5.5 Proof of Theorem 2

Theorem 2 is obtained using Theorem 1 and the following lemma.

Lemma 2. *Under Assumptions $(B_1)-(B_4)$ and Assumptions (A_1), (A_2), (A_6), (A_7) and (A_8), we have that*

$$\lim_{n\to\infty}\sqrt{n\psi(h_n)}\left(\frac{E(g_n(x))}{E(f_n(x))}-\phi(x)\right)=0.$$

Proof. We have the following decomposition

$$\frac{E(g_n(x))}{E(f_n(x))}-\phi(x)=\frac{1}{E(f_n(x))}\left[E(g_n(x))-f(x)\phi(x)\right]+\frac{\phi(x)}{E(f_n(x))}\left[f(x)-E(f_n(x))\right]. \tag{11}$$

Here we study first the quantity $E(f_n(x))-f(x)$. As in Lemma 1, we have

$$E(f_n(x))=\sum_{k=1}^{m}\left(\frac{1}{\psi(h_n)}E\left(K_n(\varphi(\xi,z_k))\right)\frac{1}{n}\sum_{i=0}^{n-1}P(Z_i=z_k)\right)$$

and then

$$\begin{aligned}E(f_n(x))-f(x)=&\sum_{k=1}^{m}\left(\frac{1}{\psi(h_n)}E\left(K_n(\varphi(\xi,z_k))\right)-d_k(x)c_k(x)1_{\{k\in\Delta\}}\right)\frac{1}{n}\sum_{i=0}^{n-1}P(Z_i=z_k)\\&+\sum_{k=1}^{m}d_k(x)c_k(x)1_{\{k\in\Delta\}}\left(\frac{1}{n}\sum_{i=0}^{n-1}P(Z_i=z_k)-p_k\right).\end{aligned}$$

Under the Assumptions (B_2), (B_3) and (B_4) imply that

$$\lim_{n\to\infty}\sqrt{n\psi(h_n)}\left(E(f_n(x))-f(x)\right)=0. \tag{12}$$

Here we study the term $E(g_n(x))-f(x)\phi(x)$. It satisfies

$$|E(g_n(x))-f(x)\phi(x)|\le|\phi(x)|\,|E(f_n(x))-f(x)|+|E(g_n(x))-E(f_n(x))\phi(x)|.$$

We have

$$\begin{aligned}|E(g_n(x))-E(f_n(x))\phi(x)|&=\left|\frac{1}{\psi(h_n)}\sum_{k=1}^{m}E\left[K_n(\varphi(\xi,z_k))\left(\phi(\varphi(\xi,z_k))-\phi(x)\right)\right]\frac{1}{n}\sum_{i=0}^{n-1}P(Z_i=z_k)\right|\\&\le Lh_n^{\beta}\sum_{k=1}^{m}\frac{1}{\psi(h_n)}E\left[K_n(\varphi(\xi,z_k))\left\|\frac{\varphi(\xi,z_k)-x}{h_n}\right\|^{\beta}\right]\frac{1}{n}\sum_{i=0}^{n-1}P(Z_i=z_k)\\&\le L_1h_n^{\beta}(1+o(1)),\end{aligned} \tag{13}$$

where the second line comes from Assumption (B_1) and the third is a consequence of assertion 2) of Lemma 1. From (12), (13) and Assumption (B_3), we deduce that

$$\lim_{n\to\infty}\sqrt{n\psi(h_n)}\left(E(g_n(x))-f(x)\phi(x)\right)=0. \tag{14}$$

Assertion 1) of Lemma 1, (12) and (14) applying in (11) imply the proposition. □

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